J. Appl. Maths Mechs Vol. 55, No. 6, pp. 836–843, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00

BIORTHOGONAL EXPANSIONS IN THE FIRST FUNDAMENTAL PROBLEM OF ELASTICITY THEORY†

M. D. KOVALENKO

Moscow

(*Received* 20 *September* 1990)

The first fundamental boundary-value problem of elasticity theory is considered for a rectangular semi-infinite strip whose long sides are free of stress. Separation of variables is used to reduce the solution to a series expansion of two functions defined in a closed interval (the "end" of the half-strip), in terms of homogeneous solutions. The system of homogeneous solutions over an interval of the real axis is proved to be complete in L_2. Systems of functions biorthogonal to the systems of homogeneous solutions are constructed on a certain contour on the Riemann surface of the logarithm. This biorthogonality concept is a natural generalization of biorthogonality over a closed interval. The biorthogonal systems constructed are used to find explicit expressions for the expansion coefficients.

1. STATEMENT OF THE PROBLEM

WE WILL consider the solution of the first fundamental problem of elasticity theory in a half-strip ($|y|\leqslant 1$, $0\leqslant x<\infty$). We shall assume that the long sides of the half-strip are unstressed:

$$\sigma_y(x, \pm 1) = \tau_{xy}(x, \pm 1) = 0 \tag{1.1}$$

while the end surface $\{x=0, y\in(-1, 1)\}$ is subject to the following stresses:

$$\sigma_x(y) = \alpha(y),\ \tau_{xy}(y) = \beta(y),\ y\in(-1, 1) \tag{1.2}$$

We will confine ourselves to symmetric deformations of the half-strip. Then, in the class of solutions which decay at infinity ($x\to\infty$):

$$\int_{-1}^{1} \alpha(y)\,dy = 0$$

Using separation of variables [1], we can reduce the boundary-value problem to the expansions

$$\alpha(y) = \sum_{k=1}^{\infty} 2\operatorname{Re}(a_k\lambda_k\sigma_k(y))$$

$$\beta(y) = \sum_{k=1}^{\infty} 2\operatorname{Re}(a_k\lambda_k^2\tau_k(y)),\quad y\in(-1, 1),\quad a_k\in C \tag{1.3}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 956–963, 1991.

in which

$$\sigma_k(y) = (\sin\lambda_k - \lambda_k\cos\lambda_k)\cos\lambda_k y - \lambda_k y\sin\lambda_k\sin\lambda_k y \qquad (1.4)$$
$$\tau_k(y) = \cos\lambda_k\sin\lambda_k y - y\sin\lambda_k\cos\lambda_k y$$

The numbers $\{\lambda_k\}_{k=1}^{\infty} = \Lambda$ are all the complex zeros of the entire function

$$L(\lambda) = \lambda + \sin\lambda\cos\lambda \qquad (1.5)$$

There are a great many approximate methods of determining the unknowns $\{a_k, \bar{a}_k\}_{k=1}^{\infty}$ using the expansions (1.3). A survey of these methods may be found in [2, 3].

In this paper we will construct systems of functions $\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$ and $\varphi_\nu(\omega)_{\nu=1}^{\infty}$ which are biorthogonal, over a certain contour T in the domain of the complex variable $\omega = x+iy$, to the systems $\{\sigma_k(\omega)\}_{k=1}^{\infty}$ and $\{\tau_k(\omega)\}_{k=1}^{\infty}$, respectively. The functions $\sigma_k(\omega)$ and $\tau_k(\omega)$ $(k\geqslant 1)$ are continuations of $\sigma_k(y)$ and $\tau_k(y)$ to the ω domain. Using biorthogonal systems of functions, we can find explicit expressions for the coefficients $a_k, \bar{a}_k$ of the expansions (1.3), which we shall henceforth call biorthogonal expansions.

2. COMPLETENESS OF SYSTEMS OF REAL SUBSPACES $\{\mathrm{Re}(a_k\sigma_k(y))\}_{k=1}^{\infty}$ AND $\{\mathrm{Re}(a_k\tau_k(y))\}_{k=1}^{\infty}$

We will present a simple proof of the completeness of the system $\{\mathrm{Re}(a_k\tau_k(y))\}_{k=1}^{\infty}$ in $L_2(-1,1)$. The completeness of systems of functions similar to (1.4) has been considered, e.g. in [4, 5].

We will begin with the basic properties of the function $L(\lambda)$ defined by (1.5). These properties are easily established, e.g. using results from [6]. The function $L(\lambda)$ is entire, of completely regular growth and of exponential type 2. The indicator diagram of $L(\lambda)$ is the interval $[-2, 2]$ on the imaginary axis. Its zeros satisfy the asymptotic relation

$$\lambda_k \sim \pm\left(k\pi - \frac{\pi}{4}\right) \pm \frac{i}{2}\ln 4k\pi \quad (k\to\infty)$$

Theorem 1. The system of real subspaces $\{\mathrm{Re}(a_k\tau_k(y))\}_{k=1}^{\infty}$ $(a_k\in C)$ is complete in $L_2(-1,1)$.

Proof. Let $\tau(\lambda, y)$, $\lambda\in C$, $\mathrm{supp}\,\tau(\lambda,y)\in[-1,1]$ be the function generating the system $\{\tau_k(y)\}_{k=1}^{\infty}$ for $\lambda\in\Lambda$; let $a(\lambda)$ be any function such that $a_k = a(\lambda_k)$. We will first prove that $\tau(\lambda, y)$ is a closed kernel in $L_2(-1,1)$, i.e. there is no compactly supported function $\chi(y)\in L_2(-1,1)$ which is not equivalent to zero and has the property

$$\int_{-1}^{1}\mathrm{Re}(a(\lambda)\tau(\lambda, y))\chi(y)\,dy = 0, \quad \lambda\in C \qquad (2.1)$$

Since $a(\lambda)$ is arbitrary, this is possible if

$$\int_{-1}^{1}\tau(\lambda, y)\chi(y)\,dy = 0, \lambda\in C \qquad (2.2)$$

Solving Eq. (2.2), we find that $\chi(y) = c[\delta(y+1) - \delta(y-1)]$ [c is an arbitrary constant and $\delta(\cdot)$ is the delta-function], so that $\tau(\lambda, y)$, and hence also $\mathrm{Re}(a(\lambda)\tau(\lambda, y)$ is a closed kernel in $L_2(-1,1)$.

Let $\xi(y)$ $(\mathrm{supp}\,\xi(y)\in(-\gamma,\gamma)$, $0<\gamma<1)$ be a function of compact support in $L_2(-1,1)$ such that

$$\int_{-\gamma}^{\gamma} \operatorname{Re}(a_k \tau_k(y)) \xi(y)\, dy = 0, \quad k \geqslant 1 \tag{2.3}$$

Define

$$\Phi(\lambda) = \int_{-\gamma}^{\gamma} \operatorname{Re}(a(\lambda) \tau(\lambda, y) \xi(y)\, dy, \quad \lambda \in C \tag{2.4}$$

By (2.3), $\Phi(\lambda_k) = 0$ $(k \geqslant 1)$. Hence

$$P(\lambda_k) = \int_{-\gamma}^{\gamma} \tau_k(y) \xi(y)\, dy = 0, \quad k \geqslant 1 \tag{2.5}$$

It follows from (2.5) that the entire function $P(\lambda)$ is of type at least 2 [since its zeros are at least all the complex zeros of $L(\lambda)$, which is of type 2].

On the other hand, by the Paley–Wiener Theorem [7], the type of the entire function $P(\lambda)$ is at most $1+\gamma$. By the uniqueness theorem [6, 8], we obtain $P(\lambda) \equiv 0$ if $\gamma < 1$. And since $\operatorname{Re}(a(\lambda)\tau(\lambda, y))$ is a closed kernel in $L_2(-1, 1)$, a standard completeness criterion [8] implies that the system of subspaces $\{\operatorname{Re}(a_k \tau_k(y))\}_{k=1}^{\infty}$ is complete in $L_2(-1, 1)$.

The completeness of the system $\{\operatorname{Re}(a_k \sigma_k(y))\}_{k=1}^{\infty}$ is proved in a similar way.

Remark. The completeness of the systems of real subspaces $\{\operatorname{Re}(a_k \sigma_k(y))\}_{k=1}^{\infty}$ and $\{\operatorname{Re}(a_k \tau_k(y))\}_{k=1}^{\infty}$ is equivalent to double completeness of the systems $\{\operatorname{Re}\sigma_k(y), \operatorname{Im}\sigma_k(y)\}_{k=1}^{\infty}$ and $\{\operatorname{Re}\tau_k(y), \operatorname{Im}\tau_k(y)\}_{k=1}^{\infty}$.

3. GENERALIZED BOREL TRANSFORMS ON THE RIEMANN SURFACE OF THE LOGARITHM

Let $G(z)$ be a quasi-entire function, i.e. [9, 10] a univalent analytic function defined on the Riemann surface of the logarithm $K(z) = \{z = \lambda + i\xi, |\arg z| < \infty, 0 < |z| < \infty\}$. Following [10], we shall say that a quasi-entire (entire) function belongs to class $\{1, a\}$ if it is of exponential type $\leqslant a$. In addition, by analogy with entire functions, a quasi-entire function $G(z) \in \{1, 1\}$ belongs to class W if its real part is of at most power growth over the whole real axis and square summable on the positive real axis R^+.

Consider a quasi-entire function $G(z) \in W$. Let $g(\omega)$ be the Borel transform of $G(z)$. As shown in [9],

$$G(z) = \frac{1}{2\pi i} \int_C g(\omega) e^{z\omega} d\omega, \quad \operatorname{Re} z > 0 \tag{3.1}$$

where C is a contour in the domain $\Omega = \{\omega = x + iy, |\arg\omega| \leqslant \pi, 0 < |\omega| < \infty\}$ on the Riemann surface $K(\omega) = \{\omega = x + iy, |\arg\omega| < \infty, 0 < |\omega| < \infty\}$. The contour C is formed by rays $\{L^{\pm}: re^{\pm i\pi}, r > 1 + \eta, \eta > 0\}$ and the circular arc $\{C_{1+\eta}: |\omega| = (1+\eta)e^{i \arg\omega}, |\arg\omega| \leqslant \pi\}$. It can be shown that if $G(z) \in W$, then the arc $C_{1+\eta}$ can be contracted to a rectangular contour Π enclosing the interval $[-1, 1]$ of the imaginary axis, consisting of the vertical intervals $\{l: x = \varepsilon, y \in [-1-\eta, 1+\eta]\}$, $\{l^+: x = -\varepsilon, y \in [0, 1+\eta]\}$, $\{l^-: x = -\varepsilon, y \in [-1-\eta, 0]\}$ and the horizontal intervals $\{y = \pm\eta, x \in [-\varepsilon, \varepsilon]\}$. An analogue of this assertion is included in the Plancherel–Polyá proof of the Paley–Wiener Theorem [6]. Denote the contour formed by the rays $L^{\pm}$ and the reactangle Π by T.

Let $f(y)$ be an arbitrary compactly supported function in $L_2(\Gamma)$ with support in $\{\Gamma: y \in (-1, 1)\}$.

By the Paley–Wiener Theorem, its Fourier transform $F[f](\xi)$ is an element of W [7].

Let $f(\omega)$ $(\omega = x+iy)$ be the Borel transform of $F[f]$. By [6, 8]:

$$f(\omega) = \int_0^\infty F[f](\xi)e^{-\xi\omega}d\xi, \quad \xi = te^{-i\theta}, \quad t \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi \tag{3.2}$$

and the integral exists in the half-plane $\mathrm{Re}(\omega e^{i\theta}) > h(-\theta)$, where $h(-\theta)$ is the growth indicator of $F[f]$. All the singularities of $f(\omega)$ lie in the interval Γ on the imaginary axis. Thus, formula (3.2) associates with any compactly supported function $f(y) \in L_2(\Gamma)$ a function $f(\omega)$ which is analytic in the domain $\Delta\backslash\overline{\Gamma}$.

Take $f(y) = \eta(y)\cos\lambda y$, where $\eta(y)$ is the characteristic function of Γ[1]. Denote the Borel transform of the entire function $F[\eta(y)\cos\lambda y](\xi)$ by $C(\lambda, \omega)$.

It follows from the Cauchy representation [11]

$$C(\lambda, \omega) = \int_\Gamma \frac{\cos\lambda y}{iy-\omega}dy, \quad \omega \in \Omega\diagdown\overline{\Gamma}, \quad \lambda \in C \tag{3.3}$$

and the Paley–Wiener Theorem that $C(\lambda, \omega)$ is an entire function of the parameter λ in the class W. It is also obvious that $C(\lambda, \infty) = 0$.

Proposition 1. Let $g(\omega)$ be a function analytic on the Riemann surface of the logarithm $K(\omega)$, all of those sheets are cut along the intervals $[-1, 1]$ of the imaginary axis and $g(\infty) = 0$. If moreover

$$g(y) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}[g(iy+\varepsilon) - g(iy-\varepsilon)] \in L_2(\Gamma)$$

then the function

$$G(z) = \frac{1}{2\pi i}\int_\Gamma g(\omega)C(z, \omega)d\omega \tag{3.4}$$

is holomorphic in the domain $Z = \{z + i\zeta, \ |\arg z| < \pi, \ 0 < |z| < \infty\}$. The analytic continuation of $G(z)$ to $K(z)$ is a quasi-entire function of class W.

Proof. We will outline the proof. The existence of the integral (3.4) is obvious. It follows from representation (3.3) for $C(z, \omega)$ that the function $G(z)$ exists in the domain Z together with all its derivatives, i.e. it is analytic in Z. And since $C(z, \omega) \in \{1, 1\}$ and the integral (3.4) is absolutely convergent, it follows that $G(z) \in \{1, 1\}$.

We will show that $G(z)$ is square summable over the positive real axis R^+. Contract the contour Π to the interval $\overline{\Gamma}$ of the imaginary axis. Then

$$\frac{1}{2\pi i}\int_\Pi C(z, \omega)g(\omega)d\omega = \int_\Gamma g(y)\cos zy\,dy \tag{3.5}$$

Taking into consideration that $g(y) \in L_2(\Gamma)$ (by the Paley–Wiener Theorem), we conclude that the integral (3.5) is an entire function in the class W. On the rays $L^\pm$ we have $G(\lambda) \in L_2(R^+)$, because $C(\lambda, \omega) \in W$.

Thus $G(z)$ is analytic in Z and of class W. It remains to prove that $G(z)$ admits of analytic continuation to the Riemann surface $K(z)$, i.e. it is quasi-entire. This is easily done by well-known means [9, 10].

By analogy with the case of entire function [8], we shall say that $g(\omega)$ is $C(z, \omega)$-associated with the quasi-entire function $G(z)$, and (3.4) will be called the generalized Borel integral transform of $g(\omega)$ on the Riemann surface of the logarithm.

The proof of the following proposition is based on a method used in [12] to construct functions which are biorthogonal to certain generalizations of systems of exponential functions.

Proposition 2. To every quasi-entire function $G(z)\in W$ there corresponds a unique function $g(\omega)$, regular on the contour T and in its exterior, such that (3.4) is true.

Proposition 3. Let $H(z)$ and $G(z)$ be an entire and a quasi-entire function, respectively, in class W, and $h(\omega)$ and $g(\omega)$ functions $C(z,\omega)$-associated with them. Then the following Parseval-type identity holds:

$$\frac{1}{2\pi i}\int_T g(\omega)\overline{h(\omega)}\,d\omega=\frac{1}{\pi}\int_0^\infty G(\lambda)\overline{H(\lambda)}e^{-2\varepsilon\lambda}\,d\lambda,\quad \varepsilon\geqslant 0 \tag{3.6}$$

Proof. The existence of the integral along T is obvious. The integral on the right also exists since by assumption $G(\lambda), H(\lambda)\in L_2(R^+)$.

"Stretch" the contour Π along the imaginary axis, downward and upward, to infinity [that this may be done follows from the analyticity of $g(\omega)$ and $h(\omega)$ outside T]. Denote the extension of the segment l to $\pm\infty$ by l_∞ and the extension of the segments $l^\pm$ to $+\infty$ and $-\infty$, respectively, by $l_\infty{}^\pm$. By the Cauchy residue theorem, the integrals over the unions of the straight lines $l_\infty{}^+\cup L^+$ and $l_\infty{}^-\cup L^-$ vanish, and consequently

$$\frac{1}{2\pi i}\int_T g(\omega)\overline{h(\omega)}\,d\omega=\frac{1}{2\pi i}\int_{l_\infty} g(iy+\varepsilon)\overline{h(iy+\varepsilon)}\,d(iy+\varepsilon),\quad \varepsilon>0 \tag{3.7}$$

On the other hand, using the representation (3.3) of $C(\lambda,\omega)$, we obtain

$$\frac{1}{\pi}\int_0^\infty H(\lambda)C(\lambda,\omega)\,d\lambda=h(\omega),\quad \omega\in\Omega\diagdown\overline{\Gamma} \tag{3.8}$$

Substituting (3.8) into the right-hand side of (3.7) and performing some simple algebra (as was done in [11]), we obtain (3.6).

4. BIORTHOGONAL SYSTEMS OF FUNCTIONS

Let $\sigma_k(\omega)$ be the functions corresponding to the compactly supported functions $\sigma_k(y)$ $(k\geqslant 1)$ as in (3.2). Obviously,

$$\sigma_k(\omega)=(\sin\lambda_k-\lambda_k\cos\lambda_k)\,C(\lambda_k,\omega)+\lambda_k\sin\lambda_k\frac{d}{d\lambda_k}(C(\lambda_k,\omega)),\qquad \omega\in\Omega\diagdown\overline{\Gamma},\quad k\geqslant 1. \tag{4.1}$$

Let $\{\psi_\nu(\omega)\}_{\nu=1}^\infty$ be a system of functions analytic on and in the exterior of T, with $\psi_\nu(\infty)=0$, $\nu\geqslant 1$. The function

$$\sigma(\lambda,\omega)=(\sin\lambda-\lambda\cos\lambda)\,C(\lambda,\omega)+\lambda\sin\lambda\frac{d}{d\lambda}(C(\lambda,\omega)),\quad \lambda\in C,\quad \omega\in\Omega\diagdown\overline{\Gamma} \tag{4.2}$$

generates the system $\{\sigma_k(\omega)\}_{\nu=1}^\infty$ for $\lambda\in\Lambda$.

Suppose that the following equality holds on the positive real axis $\lambda\in R^+$:

$$\frac{1}{2\pi i}\int_T \sigma(\lambda,\omega)\psi_\nu(\omega)\,d\omega=\frac{\lambda^2L(\lambda)}{(\lambda^2-\lambda_\nu^2)(\lambda^2-\bar\lambda_\nu^2)}=R_\nu(\lambda),\quad \nu\geqslant 1 \tag{4.3}$$

The interval (4.3) exists in Z. This follows from the representation (4.2) of $\sigma(\lambda,\omega)$ and

Proposition 1. Since the left-hand and right-hand sides of (4.3) are entire functions, the validity of these equalities for $\lambda \in R^+$ implies their validity throughout Z. Then, setting $\lambda = \lambda_k$, $\lambda_k \in \Lambda$ in (4.3), we arrive at

$$\frac{1}{2\pi i}\int_T \sigma_k(\omega)\psi_\nu(\omega)\,d\omega = \begin{cases} N_k = R_k(\lambda_k), & k=\nu \\ 0, \quad k \neq \nu & (k, \nu \geqslant 1) \end{cases} \tag{4.4}$$

A system of functions $\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$ satisfying (4.4) is said to be biorthogonal to the system $\{\sigma_k(\omega)\}_{k=1}^{\infty}$.

Set

$$\Psi_\nu(\lambda) = \frac{1}{2\pi i}\int_T \psi_\nu(\omega)\,C(\lambda,\omega)\,d\omega, \qquad \lambda \in R^+, \qquad \nu \geqslant 1 \tag{4.5}$$

Substituting (4.2) into (4.3), we obtain the following equations for the functions $\Psi_\nu(\lambda)$:

$$(\sin\lambda - \lambda\cos\lambda)\Psi_\nu(\lambda) + \lambda\sin\lambda\, d\Psi_\nu(\lambda)/d\lambda = R_\nu(\lambda),\ \lambda \in R^+,\ \nu \geqslant 1 \tag{4.6}$$

A particular solution of these equations may be written as

$$\Psi_\nu(\lambda) = \frac{\sin\lambda}{\lambda}\int_0^\lambda \frac{R_\nu(\lambda)\,d\lambda}{\sin^2\lambda}, \qquad \nu \geqslant 1 \tag{4.7}$$

Hence, using the Mittag–Leffler expansion [13] of the meromorphic function in the integrand, we obtain

$$\Psi_\nu(\lambda) = -\sum_{n=1}^{\infty}\frac{R_\nu(p_n)\,\lambda\sin\lambda}{p_n(\lambda^2 - p_n^2)} + \sum_{n=1}^{\infty}\frac{r_\nu(p_n)\ln|1-\lambda^2 p_n^{-2}|\sin\lambda}{\lambda} \tag{4.8}$$

$$r_\nu(p_n) = \frac{d}{d\lambda}(\lambda^{-2}R_\nu(\lambda))|_{\lambda=p_n}, \qquad p_n = n\pi, \qquad \lambda \in R^+, \qquad \nu \geqslant 1$$

Using bounds $|R_\nu(p_n)|$, $|r_\nu(p_n)|$, one can show that the series (4.8) are uniformly convergent.

Let $S_{1\nu}(\lambda)$, $S_{2\nu}(\lambda)$, $(\nu \geqslant 1)$ be the sums of the first and second series in (4.8), respectively. We will first consider the second sum $S_{2\nu}(\lambda)$. The analytic continuation of each term of $S_2(\lambda)$ (henceforth we will omit the subscript ν) is a quasi-entire function in class W. But since the series $S_2(\lambda)$ is uniformly convergent, the analytic continuation of its sum $S_2(z)$ is a quasi-entire function in W [9]. The function $S_2(z)$ may be expressed as

$$S_2(z) = Q(z)\ln z, \quad Q(z) \in W \tag{4.9}$$

(this follows from the fact that after the substitution $\lambda = \pm p_n(1-u)$, $(n \geqslant 1)$ each term of $S_2(\lambda)$ can be reduced to this form), and hence this function is defined on the Riemann surface $K(z)$.

Now consider the sum of the first series $S_1(\lambda)$ in (4.8). Since each term of the series is an entire function in class W and the series itself is uniformly convergent, $S_1(\lambda) \in W$.

Let $\Psi_\nu(z)$ $(\nu > 1)$ be the analytic continuation of the functions $\Psi_\nu(\lambda)$ to $K(z)$. As just shown, such a continuation exists and is the sum of an entire function and a quasi-entire function in class W. By Proposition 2, the existence of the system of functions $\{\Psi_\nu(z)\}_{\nu=1}^{\infty}$ implies the existence of the $C(z,\omega)$-associated system $\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$, which satisfies Eqs (4.3), i.e. it is biorthogonal to the system $\{\sigma_k(\omega)\}_{k=1}^{\infty}$.

The uniqueness of the biorthogonal system is proved as follows. The system of functions

$\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$ is not unique if the right-hand side of (4.3) can be multiplied by an entire function of zero type with no zeros (so as not to affect the completeness of the system of functions $\{\mathrm{Re}(a_k\sigma_k(y))\}_{k=1}^{\infty}$). By the Phragmen–Lindelöf Theorem [6], the only functions meeting these requirements are constants.

The arguments presented above constitute the content of the following theorem.

Theorem 2. There exists a unique system of functions $\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$ which is biorthogonal to the system $\{\sigma_k(\omega)\}_{k=1}^{\infty}$ in the sense of (4.4).

A similar construction yields a system of functions $\{\varphi_\nu(\omega)\}_{\nu=1}^{\infty}$ biorthogonal on T to the system $\{\tau_\nu(\omega)\}_{k=1}^{\infty}$. The functions $\varphi_\nu(\omega)$ are defined by the equations

$$\frac{1}{2\pi i}\int_T \tau(\lambda,\omega)\,\varphi_\nu(\omega)\,d\omega = R_\nu(\lambda), \qquad \lambda \in R^+, \qquad \nu \geqslant 1$$

$$\tau(\lambda,\omega) = \cos\lambda S(\lambda,\omega) - \sin\lambda\,\frac{d}{d\lambda}(S(\lambda,\omega)) \tag{4.10}$$

Here $S(\lambda, \omega)$ is the Borel transform of the entire function $F[\eta(y)\sin\lambda y]$.

5. BIORTHOGONAL EXPANSIONS

Using the biorthogonal systems $\{\psi_\nu(\omega)\}_{\nu=1}^{\infty}$ and $\{\varphi_\nu(\omega)\}_{\nu=1}^{\infty}$, we find the coefficients $a_k, \bar{a}_k$ $(k\geqslant 1)$, of expansions (1.3). to that end, we consider the Fourier transforms of (1.3) and then, using (3.2), obtain equalities for the Borel transforms. Multiplying the first of these equalities by $\psi_\nu(\omega)$ and the second by $\varphi_\nu(\omega)$, integrating along T and using (4.4) and the analogue of the latter for the system $\{\varphi_\nu(\omega)\}_{\nu=1}^{\infty}$, which follows from (4.10), we obtain a system of two algebraic equations for each $\nu \geqslant 1$ in the unknowns $a_\nu, \bar{a}_\nu$:

$$\alpha_\nu = 2\,\mathrm{Re}\,(a_\nu\lambda_\nu N_\nu), \quad \beta_\nu = 2\,\mathrm{Re}\,(a_\nu\lambda_\nu^2 N_\nu) \tag{5.1}$$

and, by (3.6),

$$\alpha_\nu = \frac{1}{2\pi i}\int_T \psi_\nu(\omega)\,\alpha(\omega)\,d\omega = \frac{1}{\pi}\int_0^\infty \Psi_\nu(\lambda)\,F[\alpha](\lambda)\,e^{-2\varepsilon\lambda}\,d\lambda$$

$$\beta_\nu = \frac{1}{2\pi i}\int_T \varphi_\nu(\omega)\,\beta(\omega)\,d\omega = \frac{1}{\pi}\int_0^\infty \Phi_\nu(\lambda)\,F[\beta](\lambda)\,e^{-2\varepsilon\lambda}\,d\lambda, \qquad \varepsilon \geqslant 0 \tag{5.2}$$

$$\left(\Phi_\nu(\lambda) = \frac{1}{2\pi i}\int_T \varphi_\nu(\omega)\,S(\lambda,\omega)\,d\omega, \qquad \nu \geqslant 1, \qquad \lambda \in R^+\right)$$

Here $\alpha(\omega)$ is the function $C(\lambda, \omega)$-associated with $F[\alpha](\lambda)$ and $\beta(\omega)$ is the function $S(\lambda, \omega)$-associated with $F[\beta](\lambda)$.

Example. We will give a simple example of biorthogonal expansions (1.3). Take $\alpha(y) = 1/3 = y^2$, $\beta(y) = 0$. Obviously, $\beta_\nu = 0$ $(\nu \geqslant 1)$. Taking into account that $\lim_{\lambda\to 0}\lambda^{-3}\sigma(\lambda, y) = 1/3 - y^2$, we deduce from (4.3), letting $\lambda \to 0$, that $\alpha_\nu = 2/|\lambda_\nu^2|$. Now, solving the system of equations (5.1), we obtain

$$a_\nu = \frac{2}{|\lambda_\nu^2|}\cdot\frac{\bar{\lambda}_\nu}{N_\nu\lambda_\nu(\lambda_\nu - \bar{\lambda}_\nu)}$$

The correctness of this solution has been verified by inserting the computed values of the coefficients a_ν into the series (1.3); it turns out that by retaining 25 terms and summing one obtains the limiting function with an error not exceeding 3%.

REFERENCES

1. VLADIMIROV V. S., *The Equations of Mathematical Physics*. Nauka, Moscow, 1981.
2. GRINCHENKO V. T., *Equilibrium and Steady Vibrations of Elastic Bodies of Finite Size*. Naukova Dumka, Kiev, 1978.
3. PROKOPOV V. K., Homogeneous solutions of elasticity theory and their applications to thin plate theory. In *Proceedings of the 2nd All-Union Conference on Theoretical and Applied Mechanics: Solid State Mechanics*, pp. 253–259. Nauka, Moscow, 1966.
4. USTINOV Yu. A. and YUDOVICH V. I., On the completeness of a system of elementary solutions of the biharmonic equation in a half-strip. *Prikl. Mat. Mekh.* **37,** 4, 706–714, 1973.
5. VOROVICH I. I. and KOVAL'CHUK V. E., On the basis properties of a system of homogeneous solutions. *Prikl. Mat. Mekh.* **31,** 5, 861–869, 1967.
6. LEVIN B. Ya., *The Distribution of the Roots of Entire Functions*. Gostekhizdat, Moscow, 1956.
7. AKHIYEZER N. I., *Lectures on Approximation Theory*. Nauka, Moscow, 1965.
8. IBRAGIMOV I. I., *Methods of Interpolating Functions and Some of Their Applications*. Nauka, Moscow, 1971.
9. PFLÜGER A., Über eine Interpretation gewisser Konvegenz- und Fortsetzungseigenschaften Dirichletscher Reihen. *Comm. Mat. Helv.* **11,** 89–124, 1935/36.
10. DZHRBASHYAN M. M., *Integral Transforms and Representations of Functions in the Complex Domain*. Nauka, Moscow, 1966.
11. BREMERMANN H. J., *Distributions, Complex Variables and Fourier Transforms*. Addison-Wesley, New York, 1965.
12. LEONT'YEV A. F., *Generalizations of Series of Exponential Functions*. Nauka, Moscow, 1981.
13. LAVRENT'YEV M. A. and SHABAT B. V., *Methods of the Theory of Functions of a Complex Variable*. Fizmatgiz, Moscow, 1958.

Translated by D.L.